

DID PEIRCE HAVE HILBERT'S NINTH AND TENTH PROBLEMS?

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Abstract. In the manuscript “Logical Studies of the Theory of Numbers” (ca. 1890), Peirce asks whether there is an algorithm for finding solutions to equations in number theory. He thus anticipates, in a more general way, David Hilbert’s Tenth Problem, posed at the International Congress of Mathematicians in 1900, of determining whether there is an algorithm for solutions to Diophantine equations. Peirce proposes translating these equations into Boolean algebra, but does not show how to use that to solve equations. In 1931 Gödel proved that number theory is incomplete, and in 1970 Yuri Matiyasevich gave a negative answer to Hilbert.

Sommaire. Dans le manuscrit «Logical Studies of the Theory of Numbers», Peirce demande s’il existe un algorithme pour trouver des solutions aux équations en théorie des nombres. Il a donc anticipé, d’une manière plus générale, dixième problème de David Hilbert, posée au Congrès International de mathématiques en 1900, de déterminer s’il existe un algorithme pour trouver des solutions aux équations diophantienne. Peirce propose de traduire ces équations en algèbre booléenne, mais n’affiche pas comment l’utiliser pour résoudre des équations. En 1931 Gödel s’est avéré que la théorie des nombres est incomplète, et en 1970, Yuri Matiyasevich a donné une réponse négative de Hilbert.

Zusammenfassung. In das Manuskript “Logical Studies of the Theory of Numbers” fragt Peirce, ob es sich um ein Algorithmus für die Suche nach Lösungen für Gleichungen in Zahlentheorie. Er rechnet damit, auf eine allgemeinere Weise David Hilbert zehnten Problem, bei der Internationalen Kongress der Mathematiker im Jahr 1900, der bei der Beurteilung, ob es ein Algorithmus für Lösungen für diophantische Gleichungen gestellt. Peirce schlägt vor, diese Gleichungen in Boole’schen Algebra übersetzen, aber wie, die zum Lösen von Gleichungen verwendet nicht angezeigt. 1931 Bewiesen Gödel, dass Zahlentheorie unvollständig ist, und 1970 Yuri Matiyasevich eine negative Antwort auf Hilbert gab.

MSC-class: 03-03, 03B25, 11U05 (primary); 01A55, 11-03, 03D35, 03D80 (secondary)

Key words: decidability; number theory

In the manuscript “Logical Studies of the Theory of Numbers” (ca. 1890) [Peirce ca. 1890], Charles Sanders Peirce (1839–1914) asks whether there is an algorithm for finding solutions to equations in number theory. He also asks whether there is an algorithm for determining if there are proofs in number theory. “The object of the present investigation,” he writes [Peirce 2010, p. 55], “is to analyze carefully the logic of the theory of numbers. I especially desire to clear up the question of whether there can be fundamentally different ways of proving a theorem from given premises; and the law of reciprocity seems to be instructive in this respect. I also wish to know whether there is not a regular method of proof in higher arithmetic, so that we can see in advance precisely how a given proposition is to be demonstrated.” He thus seems to anticipate, in a more general way, David Hilbert’s Tenth Problem, posed at the International Congress of Mathematicians in 1900, of determining whether there is an algorithm for solutions to Diophantine equations. Peirce proposes translating these equations into Boolean algebra, but does not show how to use that to solve equations.

Like much of his work, Peirce failed to follow through with this project. The manuscript was found interpolated into an unsent letter to an unknown correspondent.

In this manuscript, Peirce (1839–1914) wants to know if there exists a procedure for solving equations in number theory. He does not specify in particular the solution of Diophantine equations, that is of polynomial equations of any number of unknowns, whose solutions are to be determined (or, for example, finding Mersenne numbers) in particular. He appears to be working with generalized polynomial equations of no particular degree. For example, a linear Diophantine equation (in two variables) is an equation of the general form $ax + by = c$, where solutions with a , b , and c integers are to be determined. (For a history of Diophantine equations, see [Bashmakova

1997]. This should be supplemented by [Davis 1973] and [Davis & Hersh 1973].) The more specific, of in some way general, question being asked is whether there is a procedure for proving a theorem (equation) from its premises (axioms of number theory). The multiplicity of proofs of the Law of Reciprocity, and its increased generalizations, suggests by means of example a clue of finding whether there are procedures for solving equations and, if so, of how, if there is more than one procedure, of selecting from among them.

The first formulation of his questions asks whether there may be different ways of proving equations from given premises, and he suggests that the Law of Reciprocity might offer a hint regarding the answer to this question. The Law of Reciprocity (also known as the Law of Quadratic Reciprocity) was stated by Leonhard Euler (1707–1783), although he was unable to prove it. (The statement and a sketch of Euler’s informal argument is given in a letter to Christian Goldbach (1690–1764) of 28 August 1742 [Euler 1742]; the letter appears in [Fuss 1843, I, 143–153] and [Juškevič & Winter 1965]; an English translation of the letter and a formal proof in modern terminology and notation is given in [Edwards 1983].) Also called the *aureum theorema* (golden theorem) by Carl (Karl Friedrich) Gauss (1777–1855), it states that: If p and q are distinct odd primes, then the congruences

$$\begin{aligned}x^2 &\equiv q \pmod{p} \\x^2 &\equiv p \pmod{q}\end{aligned}$$

are both solvable or both unsolvable unless both p and q leave the remainder 3 when divided by 4 (in which case one of the congruences is solvable and the other is not). Written symbolically,

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4}$$

where

$$\frac{p}{q} \equiv \begin{cases} 1, & \text{for } x^2 \equiv p \pmod{q} \text{ solvable for } x \\ -1, & \text{for } x^2 \equiv p \pmod{q} \text{ not solvable for } x \end{cases}$$

is known as a *Legendre symbol*, from Legendre, who gave the first complete formulation of the Law and expressed it in terms of the Legendre symbol, due to Adrian Marie Legendre (1752–1833).

Euler stated the theorem in 1783 without proof (see [Edwards 1988]). Legendre first set it forth in “Recherches d’analyse indéterminée” [Legendre 1785] and was the first to publish a proof, in his *Essai sur la théorie des nombres* [Legendre 1798], but it was fallacious. (See [Pieper 1997] and [Weintraub 2009] on Legendre’s work on the law.) In 1796, Gauss became the first to publish a correct proof ([Gauss 1801, §41, 141–145; see esp. p. 144; see [Frei 2007, 180] for the account, for based on Gauss’s diary; Gauss’s proofs of his “Fundamental Theorem”, the Quadratic Reciprocity Law, were presented in sect. 4 of his *Disquisitiones arithmeticae* [Gauss 1801]. For details of the history of the Law, see [Lemmermeyer 2000].) The quadratic reciprocity theorem was Gauss’s favorite theorem from number theory, and he devised no fewer than eight different proofs of it over his lifetime. Another proof was added by Ferdinand Gotthold Max Eisenstein (1823–1852) [Eisenstein 1844]. (For more see [Edwards 1982, pp. 31–43] and [Laubenbacher & Pengelley 1994]). By the mid-1970s, there were several hundred proofs of the Law of Quadratic Reciprocity (see, e.g. [Adams & Goldstein 1976, 12]. Meanwhile, Ernest Eduard Kummer (1810–1893) [Kummer 1847a, 1847b] generalized the law by introducing ideal numbers into the field of complex numbers after his efforts to solve Fermat’s Last Theorem led him to recognize that attempts to prove the theorem broke down because the unique factorization of integers did not extend to other rings of complex numbers. He thus attempted to restore the uniqueness of factorization by introducing ideal numbers. Not only has his work been most fundamental in work relating to Fermat’s Last Theorem, since all later work was based on it for many years, but the concept of an ideal allowed ring theory, and much of abstract algebra, to develop.

The genus theorem states that the Diophantine equation

$$x^2 + y^2 = p$$

can be solved for p a prime if and only if $p \equiv 1 \pmod{4}$ or $p = 2$.

It was during this period that Peirce was writing definitions for the *Century Dictionary*, and among the definitions for which he was responsible was the Law of Reciprocity. In the *Dictionary*, Peirce defines “quadratic reci-

procity”, as a subentry of “quadratic” as: “the relation between any two prime numbers expressed by the law of reciprocity” [Peirce 1889-1910c], and refers the reader to the definition of “law”, with a subentry on the “law of reciprocity of prime numbers” which reads [Peirce 1889-1910a]: “the proposition that if p and q are two prime numbers, then, if p is a quadratic residue of q , q is also a quadratic residue of p , unless both leave the remainder 3 when divided by 4, when, if p is a quadratic residue of q , then q is not a quadratic residue of p .” Although these articles are unsigned, we are certain that these definitions were among his contributions.¹

The subject of the multiplicity of proofs of the Law of Quadratic Reciprocity was, as a matter of fact, the subject of discussion during the period when that issue was raised by Peirce. For example, in his review of the new textbook on number, *Die Elemente der Zahlentheorie* [Bachmann 1892] by Paul Gustav Heinrich Bachmann (1837–1920) for the *Bulletin of the New York Mathematical Society*, Jacob William Albert Young (1865–1948) wrote [Young 1894, 218–220]:

SECTION III. *Quadratic Residues*.—Quadratic residues are defined, Euler’s criterion for the possibility of the congruence $x^n \equiv n \pmod{p}$ is set up, Legendre’s symbol is introduced, the congruence $x^n \equiv \pmod{m}$ is considered for composite moduli of various forms, and then the problem is attacked: “To determine of what moduli a given number n is residue (or non-residue)” Restricting consideration first to prime moduli, we ask successively, “Of what odd primes is the numbers, — 1, 2, any other prime q , residue (or non-residue)?” For the first two cases the question is satisfactorily and readily answered, while the third leads to Legendre’s *law of reciprocity* of quadratic residues. The consideration of this important theorem is begun by proving the “Gaussian lemma,” that $\left(\frac{n}{p}\right) = (-1)^\mu$, where μ is the number of negative numbers in the series of absolutely least residues \pmod{p} of

$$n, 2n, 3n, \dots, \frac{p-1}{2}n.$$

Before proceeding to a proof of the law of reciprocity, the author classifies the proofs which have been given into four categories:

First Category.—The first proof of Gauss, simplified by Dirichlet, constitutes a category by itself. Of all the proofs ever given, it attacks the problem most directly, and by induction achieves the result through simple conclusions from the idea of a quadratic residue.

Second Category.—The proofs of this category are based upon the theory of quadratic forms. To this category belong the second proof of Gauss, those of Kummer, and that of Legendre (incomplete).

Third Category.—Proofs directly connected with the division of the circle. They are exemplified by the fourth and the sixth proof of Gauss.

Fourth Category.—Proofs based on the Gaussian lemma. Examples, the third and the fifth proof of Gauss.

As the first category is treated in full in Dedekind-Dirichlet’s text-book, and as the second and the third category are not elementary, they are (with one exception in the second category to be noticed later) excluded from detailed consideration.

Of the fourth category, the proof of the pastor Zeiler is given.* This proof, though simple and elegant, has not before, as far as the reviewer knows, been presented in a textbook on the theory of numbers.† There follow Jacobi’s generalization of Legendre’s symbol and the generalized law of reciprocity, Eisenstein’s algorithm for the determination by $\left(\frac{m}{n}\right)$, m and n being any two relatively prime integers of which at least one is positive, while with Schering’s and Kronecker’s generalization of the Gaussian lemma and, based on it, Schering’s proof of the generalized law of reciprocity, and Kronecker’s representation of the symbol $\left(\frac{Q}{P}\right)$, (Q and P being any two positive, odd, relatively prime integers) as the sign of certain products and the consequent simplification of Gauss’ third proof, and as an application of Kronecker’s representation another proof by Kronecker of the law of reciprocity, bring us into the field of the most recent researches in this subject.

* *Berl. Monatsber.* 1872.

† It may be of interest to give in this connection a list of the proofs presented in various text-books:

Tscheytscheff, Gauss III.

Dedekind-Dirichlet, Gauss I, III, IV.

Serret (Alg. Sup.), Gauss III.

Bachmann (Kreistheilung), $\left\{ \begin{array}{l} \text{Gauss VI, Eisenstein (Crelle 27).} \\ \text{Liouville, Eisenstein (Crelle 29).} \\ \text{Gauss IV (sketch).} \end{array} \right.$

Wertheim, Gauss V, Eisenstein (*Crelle* 28).

Mathews, Gauss I, III, Eisenstein (*Crelle* 27, in part).

Peirce was a member of the mathematical society and there is no reason to think that he did not read these lines. What remains problematic are: (a) whether he may have written about the multiplicity of proofs of the Law before or after reading Young's remarks; and (b) whether, regardless of when, or even if, he read Young's remarks, he was not already aware of the multiplicity of proofs, and, (c), if so, he considered it in some respect to be a difficulty. These issues are problematic to the extent that the manuscript "Logical Studies of Number Theory" [Peirce *ca.* 1890] is dated only approximately, the best current approximation being *circa* 1890, primarily because the characteristics of the paper on which the manuscript was written was used by Peirce largely, if not entirely, in 1890. What we do know from the archival evidence is that the manuscript "Logical Studies of the Theory of Numbers" [Peirce *ca.* 1890] was extracted from an undated letter too an unknown correspondent, that the datation of *ca.* 1890 officially assigned by the Peirce Edition Project is only approximate, but that Peirce apparently was also reading Moritz Benedikt Cantor's (1829–1920) history of mathematics at about that time. In fact, of course the second volume of the first edition of Cantor's *Vorlesungen über Geschichte der Mathematik* first appeared in print in 1892, and it is there [Cantor 1892, 772–773], and in turn referencing Charles Henry (1859–1926) [Henry 1879] and Paul Tannery (1843–1904) [Tannery 1883], that Fermat's Last Theorem is discussed, in connection with Fermat's study of Diophantos of Alexandria (*ca.* 200–*ca.* 284). Thus, we can at most assume, with respect to the chronological reference for Peirce's composition of his manuscript, that it was composed some time very early in the decade of the 1890s, and is more likely, then, to have been inspired by a reading of Cantor's remarks on Fermat's Last Theorem than by the questions posed by David Hilbert (1862–1943) at the International Congress of Mathematicians in Paris in July 1900, which did not appear in print until 1901 and in English until the middle of 1902.

The history of the Law of Reciprocity and of the multiple proofs of it given by Gauss led Peirce to the next formulation of his question: "whether there is not a regular method of proof in the higher arithmetic", i.e. number theory, "so that we can see in advance precisely how a given proposition is to be demonstrated." Thus, Peirce seems to be asking whether there is an algorithm for determining whether (and if so, how) to determine for any given Diophantine equation, whether that equation has a solution in the integers.

Diophantine equations are generalizations of the Pythagorean formula, and have the form

$$a^n + b^n = c^n$$

where a , b , c , and n are all integers. Pierre de Fermat (1601–1665), in the margin of his copy of the *Arithmetica* of Diophantos of Alexandria (*ca.* 200–*ca.* 280 A.D.), wrote against this that he had a proof that there are no integers for n greater than 2 such that there is a solution in the positive integers to the Diophantine equation. Thus Fermat (at [Diophantus 1621, 85, problem II.8]: "Cubum autem in duos cubos, aut quadrato-quadratum in duos quadrato-quadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est dividere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet."² This has come down to us as Fermat's Last Theorem. Whereas the result had been proven over the nearly 350 years since Fermat stated his theorem for specific values of n , it was not until Andrew Wiles (b. 1953), with the aid of Richard Taylor (b. 1962) for a special recalcitrant case involving Hecke algebras [Taylor & Wiles 1995], gave his proof that a result, known as the Taniyama-Shimura (and sometimes the Taniyama-Shimura-Weil) Conjecture, which was known to imply Fermat's Last Theorem, has there been a proof of the general theorem [Wiles 1995]. (For a sketch accessible to mathematicians who are not necessarily algebraic number theorists, see [Faltings 1995].) The proof involves results in number theory, in particular algebraic number theory, and especially the theory of elliptical functions, that were only developed over the course of those three hundred fifty years since Fermat, many of those only in the twentieth century, and almost all of them by mathematicians working at attempts to prove Fermat's theorem.

The Law of Quadratic Reciprocity, like Fermat's Last Theorem, is Diophantine. Moreover, the concepts involved contribute to our consideration of various Diophantine equations. In particular, we can use the Law of Quadratic Reciprocity to determine for specific Diophantine equations which are and which are not solvable (see

[Adams & Goldstein 1976, 132–135] for examples and proofs).

In 1900 at the International Congress of Mathematicians in Paris, David Hilbert listed twenty-three unsolved problems, and set the goal for twentieth-century mathematicians of solving these problems. The Ninth Problem directly concerns the Law of Reciprocity [Hilbert 1901]:

9. Beweis des allgemeinsten Reziprozitätsgesetzes im beliebigen Zahlkörper

Für einen beliebigen Zahlkörper soll das Reziprozitätsgesetz der l -ten Potenzreste bewiesen werden, wenn l eine ungerade Primzahl bedeutet und ferner, wenn l eine Potenz von 2 oder eine Potenz einer ungeraden Primzahl ist. Die Aufstellung des Gesetzes, wie die wesentlichen Hilfsmittel zum Beweise desselben werden sich, wie ich glaube, ergeben, wenn man die von mir entwickelte Theorie des Körpers der l ten Einheitswurzeln {Bericht der Deutschen Mathematiker-Vereinigung über die Theorie der algebraischen Zahlkörper, Bd. IV, 1897. Fünfter Teil} und meine Theorie {Mathematische Annalen, Bd. 51 und Nachrichten der K. Ges. d. Wiss. zu Göttingen 1898} des relativ-quadratischen Körpers in gehöriger Weise verallgemeinert.

9. Proof of the most general law of reciprocity in any number field

For any field of numbers the law of reciprocity is to be proved for the residues of the l -th power, when l denotes an odd prime, and further when l is a power of 2 or a power of an odd prime.

The law, as well as the means essential to its proof, will, I believe, result by suitably generalizing the theory of the field of the l -th roots of unity, [Hilbert 1897] developed by me, and my theory of relative quadratic fields [Hilbert 1899].

It is mathematically equivalent to Peirce's expectation that the Law of Quadratic Reciprocity, having multiple proofs, can be generalized as applicable to any pair of primes.

The problem was partially solved by Emil Artin (1889–1962) by establishing the Artin reciprocity law which deals with abelian extensions of algebraic number fields [Artin 1927]. Additional contributions were made by Igor Rostislavovich Shafarevich (b. 1923), who provided the widest generalization of Reciprocity to date [Shafarevich 1950].

Hilbert's Tenth Problem asks [Hilbert 1901]:

10. Entscheidung der Lösbarkeit einer diophantischen Gleichung. Eine diophantische Gleichung mit irgendwelchen Unbekannten und mit ganzen rationalen Zahlkoeffizienten sei vorgelegt: *man soll ein Verfahren angeben, nach welchen sich mittels einer endlichen Anzahl von Operationen entscheiden lässt, ob die Gleichung in ganzen rationalen Zahlen lösbar ist.*

10. Determination of the solvability of a diophantine equation

Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: *to devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.*

and is mathematically equivalent, in its core and essence, to Peirce's more general question, of whether there exists an algorithm for finding a solution for every Diophantine.

In 1931 [Gödel 1931] Kurt Friedrich Gödel (1906–1978) arrived at his incompleteness theorems, and proved (1) that there are sentences of number theory that are true but unprovable within the axiom system in which they occur. He also proved (2) that any such consistent axiomatic system cannot prove its own consistency. This means that there are formally undecidable sentences of number theory. As examples of these Gödel explicitly mentioned three number-theoretic results, namely: Fermat's Last Theorem; the Goldbach Conjecture, proposed by Christian Goldbach (1690–1764) in a letter to Leonhard Euler of June 7, 1742, in which Goldbach states that "at least it seems that every number that is greater than 2 is the sum of three primes [Goldbach 1742; Dickson 1919-23, I, 421]; and the Riemann Hypothesis, proposed by Georg Friedrich Bernhard Riemann (1826–1866) [Riemann 1859], according to which the nontrivial Riemann zeta function zeros, i.e., the values of s other than $-2, -4, -6, \dots$ such that $\zeta(s) = 0$ (where $\zeta(s)$ is the Riemann zeta function) all lie on the "critical line" $\sigma = \mathcal{R}[s] = \frac{1}{2}$ (where $\mathcal{R}[s]$ denotes the real part of s). A more general statement known as the Generalized Riemann hypothesis conjectures that neither the Riemann

zeta function nor any Dirichlet L -series has a zero with real part larger than $\frac{1}{2}$. As yet, examples pile up for specific cases, but no definitive proof of either of these two results in their most general formulations has been obtained.

Peirce for his part followed up the first formulation of his question regarding number theory, which was mathematically equivalent to asking for a generalization of the Law of Quadratic Reciprocity, and whether there is an algorithm for dealing with equations involving quadratic residues, with a more general formulation, to ask if there is a “regular method of proof in the higher arithmetic, so that we can see in advance precisely how a given proposition is to be demonstrated.”

On the presupposition that this is a reference back to Diophantine equations specifically, this is mathematically equivalent to Hilbert’s Tenth Problem, asking whether there exists an algorithm for solving Diophantine equations in the positive integers. Although Peirce does not explicitly give an answer to this question, he proposes a method of translating number-theoretic equations into the language of the algebra of relatives. Whether recommendation of this procedure indicates on Peirce’s part a belief that the answer to the question of the existence of an algorithm for deciding the solvability of Diophantine equations, is problematic. We clearly cannot read Peirce’s mind, in particular since there is no explicit answer in the manuscript beyond an illustration of how to rewrite number-theoretic equations algebraically. Moreover, the manuscript does not offer instructions, or even a hint, of how, once a translation has been completed, the equations would be handled. It would also, therefore sheer speculation to conclude to that Peirce did not undertake, let alone complete, this task undertaken in this manuscript because he anticipated that the answer to the question would, as Yurii Vladimirovich Matiyasevich (b. 1947) [Matiyasevich 1970] proved nearly eighty years later, turn out to be negative.

Elsewhere, there is material that tends towards a preliminary conjecture that Peirce held that the “logicization” of number theory would provide an algorithm for proving all true sentences of number theory. In the manuscript “Axioms of Number” [Peirce 1880/81] and the closely related “The Axioms of Number” [Peirce ca.1881?],³ Peirce provided fifteen axioms (or assumptions) of arithmetic which provide a definition of “positive, discrete number” and from which, Peirce thought, every proposition of the theory of numbers may be deduced by formal logic, along with definitions of “addition” and “multiplication.” In related manuscripts, in particular the “Numerical Equations” [Peirce n.d.], Peirce is concerned with the method of getting all the roots when their moduli are all different, and in “Analysis of some Demonstrations concerning definite Positive Integers” [Peirce 1905-6], in which Peirce offers his existential graphs as a means of analyzing arguments of logic, and on the versos of some pages are contained notes for definitions for the *Century Dictionary*, there is a hint that Peirce was considering logic as a means of providing an algorithm for the solution of Diophantine equations.

The “Logical Studies of Number Theory” manuscript can be seen (as [Houser 2010, xliii, n. 29] suggests) as a continuation of other writings of Peirce on number theory, in particular in his efforts, in “On the Logic of Number” [Peirce 1881a] of 1881, which has been shown [Shields 1981; 1997] to be equivalent to the axiom systems for number theory of Julius Wilhelm Richard Dedekind (1831–1916) in [Dedekind 1888] *Was sind und was sollen die Zahlen* and Giuseppe Peano (1858–1932) in [Peano 1889] *Arithmetices principia, nova methodo exposita*;⁴ in the manuscript “Fundamental Properties of Number” of 5 January 1886 [Peirce 1886] in which the natural numbers are informally defined as the linear sequence of positive integers, built up by a successor operator and between which an inclusion relation either of more than and less than hold, according to which, if n has a property P , then $n + 1$ has property P if n is included in $n + 1$; and in the fragmentary manuscripts “Number” [Peirce ca. 1887a] and “Logic of Number” [Peirce ca. 1887b] of 1887, in which Peirce undertakes alternative axiomatizations of basic arithmetic, in his efforts to develop arithmetic within his algebraic logic.

There is a shift in Peirce’s approach between these previous writings and the work in “Logical Studies of Number Theory”, however. Although Peirce looks to algebraic logic formulating for the expressions of arithmetic equations, Peirce is now concerned to answer the proof-theoretic question, namely, attempting to determine whether the algebraic logic offers an algorithm for solving arithmetic equations, rather than merely a developing algebraic logic as a pasigraphy for writing out the equations and deriving them, one from the other. In short, Peirce now appears to be concerned now with using algebraic logic to develop an algorithm for numerical solutions for equations of higher arithmetic, rather than as an exercise in studying the logical structure of arithmetic equations. It may be that Peirce was stimulated in this new approach by the work he was doing for the *Century Dictionary*, and in particular for his work on the definition of “number” for the *Dictionary*.

There are two historically interesting and important anticipations which “Logical Studies of Number Theory” suggests, beyond that relating specifically to Hilbert’s Ninth Problem, then: (1) the Tenth Problem, of whether there exists an algorithm for the solution in the positive integers of Diophantine equations; and (2) the concept of encoding the syntax of one theory in the language of another theory. The second procedure, of encoding, was developed

by Gödel for his proof of the incompleteness of arithmetic. The gödelnumbers, as we know, of course, coded logical propositions of first-order logic number-theoretically, whereas Peirce’s proposal for developing, or at least searching for, an algorithm for solving number-theoretic equations went in the direction of encoding number theoretic equations in the language of Boole’s calculus—the only mathematical logic then available.

If our interpretation of what Peirce was hoping to accomplish in “Logical Studies of Number Theory”, then he must be understood as anticipating an affirmative answer to what we now know as Hilbert’s Tenth Problem. But we also know that in 1931 Gödel proved the incompleteness of arithmetic, and in particular of any axiomatic system capable of adequately formalizing number theory; and in 1970 Yuri Matiyasevich proved [Matiyasevich 1970] that the answer to Hilbert’s tenth problem is negative. It would be sheer speculation to suppose that Peirce failed to work through the details of the method he had begun to prepare to sketch because, at some point, he suspected, or even realized, that number theory is incomplete, or that there is no algorithm for solving all equations of number theory. What can, however, be said, is that Peirce inadvertently defined much of the work of mathematical logic of the twentieth century, and anticipated, if not the answers, then certainly the questions, that directed the work in mathematical logic of the twentieth century. But we still have to be careful here, even in our use of “inadvertently”, since the pivotal manuscript remained unpublished, and, so far as the history of logic is concerned, it was not Hilbert, in his Congress talk and in his publications and foundational disputes, and not Peirce, who established the goals for the direction of mathematical logic in the twentieth century.

Looking closely at Hilbert’s Tenth Problem, we see that it is concerned with the question of whether there exists an algorithm that will decide, in a finite number of steps, whether an arbitrary Diophantine equation has solutions, and, if so, how many roots (solutions) the equation has. In 1960, Martin David Davis (b. 1928), Hilary Whitehall Putnam (b. 1926), and Julia Bowman Robinson (1919–1985) proved [Davis, Putnam, & Robinson 1961], in arguments combining logic and number theory, and the following year published, their conjecture on the basis of which it is to be concluded that no such algorithm can exist. Their colleague Yuri Vladimirovich Matiyasevich (b. 1947) definitively proved that the existence of such an algorithm is impossible, demonstrating that the Davis-Putnam-Robinson Conjecture is true [Matiyasevich 1970]. (An account of the tenth problem and its solution was given by Matiyasevich [Matiyasevich 1993a], as well as in English translation [Matiyasevich 1993b]. Davis, Matiyasevich and Robinson also examined the significance of this result [Davis, Matiyasevich, & Robinson 1971]. See also [Adams & Goldstein 1976, 191n.] and [Grosswald 1984, 257]. For nontechnical expositions, see [Davis 1973].) He did so by recursively defining the Fibonacci sequence $a_0 = a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$. The short answer is that we know that Gödel’s incompleteness theorem is correct, that formal axiomatic system adequate for number theory can prove ever true statement of number theory.

We are still left with the historiographical questions of whether Peirce really was suggesting in “Logical Studies of Number Theory” that, in translating number theory into the language of formal logic, it might be possible to find such an algorithm or set of algorithms; and whether his failure to pursue the matter further arose because of a recognition, or at least an intuition, that perhaps there in fact is no such algorithm. What I suggest is that, at the minimum, Peirce did have Hilbert’s ninth and tenth problem. And, assuming that the dating of the manuscript is correct, he was at work on the problems almost a decade before they were posed by Hilbert.

Acknowledgements. This work was made possible by the cooperation and support of the members of the Peirce Edition Project, and in particular Nathan Houser for calling my attention to Peirce’s manuscript “Logical Studies of Number Theory” and for providing the impetus to pursue the issues raised by the manuscript, and Cornelis De Waal and André De Tienne for providing relevant documents and supporting the investigations of this and related manuscripts.

Notes

¹ A list of the definitions which Peirce contributed to the *Century Dictionary* is available at: <http://www.pep.éuqam.ca/listesdesmots.pep>. The Université de Montréal à Québec’s Projet d’édition Peirce (UQÀM) is selecting and preparing Peirce’s contributions to the *Century Dictionary* as vol. 7 of the *Writings of Charles S. Peirce: A Chronological Edition*. Another list, prepared by the Institute for Studies in Pragmatism, “Peirce’s *Century Dictionary* Definitions”, available at <http://www.pragmaticism.net/century.htm>, is based on [Ketner 1977].

² Quoted, [Dickson 1919-23, II, 731], [Nagell 1951, 252]; English translation on at [Ore 1988, 204]: “It is impossible to write a cube as the sum of two cubes, a fourth power as the sum of two fourth powers and in general any power beyond the

second as the sum of two similar powers. For this I have discovered a truly wonderful proof, but the margin is too small to contain it.”

³ The axioms provided in [Peirce 1880/81] are nearly identical to those which Peirce gave in his marginalia to [Peirce 1881] and manuscripts, as described in [Ketner 1977] No. 393 [Peirce 1881a], untitled, and No. 394 “Continuous, simple quantity” [Peirce 1881c]. For a detailed comparative description between [Peirce 1880/81] and [Peirce 1881b, 1881c], see [Peirce 1989], 575–576. These are the more salient manuscripts on number theory and the logic of number on which Peirce was working prior to turning to in “Logical Studies of Number Theory” [Peirce ca. 1890] to the problem of finding an algorithm for solving diophantine equations and which (see [Houser 2010, xliii, n. 28]).

⁴ The similarities between Peirce’s [1881a] and Dedekind’s [1888] axiomatic systems sufficed to precipitate a charge of plagiarism by Peirce against Dedekind; see [Gana 1985].

References

- ADAMS, William Wells, & Larry Joel GOLDSTEIN. 1976. *Introduction to Number Theory* (Englewood Cliffs, N.J.: Prentice-Hall).
- ARTIN, Emil. 1927. „Beweis des allgemeinen Reziprozitätsgesetzes“, *Abhandlungen aus dem Mathematischen Seminar der Universität* **5** (1927), 353–363.
- BACHMANN, Paul. 1892. *Die Elemente der Zahlentheorie* (Leipzig: B. G. Teubner).
- BASHMAKOVA, Izabella Grigorevna. 1997. *Diophantus and Diophantine Equations* (Washington, DC: Mathematical Association of America).
- CANTOR, Moritz Benedikt 1892. *Vorlesungen über Geschichte der Mathematik*, Bd. II: *Vom Jahre 1200 bis zum Jahre 1668* (Leipzig: B. G. Teubner).
- DAVIS, Martin. 1973. “Hilbert’s Tenth Problem is Unsolvable”, *American Mathematical Monthly* **80**, 233–269.
- DAVIS, Martin, & Reuben HERSH. 1973. “Hilbert’s 10th Problem”, *Scientific American* **229** (November), 84–91.
- DAVIS, Martin David, Yurii Vladimirovich MATIYASEVICH, & Julia Bowman ROBINSON. 1971. “Hilbert’s Tenth Problem. Diophantine Equations: Positive Aspects of the Negative Solution“, in Felix E. Browder (ed.), *Mathematical Developments Arising from Hilbert Problems* (Providence, R. I.: American Mathematical Society, 1974), 323–378; reprinted: (Solomon Feferman, ed.), *The Collected Works of Julia Robinson* (Providence, R.I: American Mathematical Society, 1996), 269–324.
- DAVIS, Martin, Hilary PUTNAM, & Julia B. ROBINSON. 1961. “The Decision Problem for Exponential Diophantine Equations”, *Annals of Mathematics* (2) **74**, 425–436.
- DEDEKIND, Richard. 1888. *Was sind und was sollen die Zahlen?* (Braunschweig: Vieweg).
- DICKSON, Leonard Eugene. 1919–23. *History of the Theory of Numbers*, 3 vols. (Washington: Carnegie Institution of Washington; New York: Chelsea Publishing Co., 1966; Mineola, N.Y.: Dover Publications, 2005).
- _____. 1919–23b. *History of the Theory of Numbers. Vol. II: Diophantine Analysis* (Washington: Carnegie Institution of Washington; New York: Chelsea Publishing Co., 1966; Mineola, N.Y.: Dover Publications, 2005), 545–550, 615–621, 731–776.
- DIOPHANTUS of Alexandria. 1621. (Claude Gaspard Bachet de Méziriac, trans., with commentary), *Diophanti Alexandrini Arithmeticon, libri sex* (Paris: Sebastiani Cramoisy, 1621; Toulouse, 1670, with notes by Pierre de Fermat).
- EDWARDS, Harold M. 1975. “The Background of Kummer’s Proof of Fermat’s Last Theorem for Regular Primes”, *Archive for History of the Exact Sciences* **1**, 219–236.
- _____. 1982. “Kummer, Eisenstein, and Higher Reciprocity Laws”, in Neal Koblitz, (ed.), *Number Theory Related to Fermat’s Last Theorem, Proceedings of the Conference sponsored by the Vaughn Foundation, Cambridge, Massachusetts, 1981* (Boston, Mass.: Birkhäuser).
- _____. 1983. “Euler and Quadratic Reciprocity”, *Mathematics Magazine* **56**, No. 5 (Nov.), 285–291.
- EISENSTEIN, Ferdinand Gotthold. 1844. „ Beweis des Reciprocitätssatzes für die cubischen Reste in der Theorie der aus dritten Wurzeln der Einheit zusammengesetzten complexen Zahlen“, *Journal für die reine und angewandte Mathematik* **27**, 289–310.
- EULER, Leonhard. 1742. Letter to Christian Goldbach, 28 August 1742; published: [Fuss 1845], I, 144–153; available online at: <http://math.dartmouth.edu/~euler/correspondence/letters/OO0768.pdf>.
- FALTINGS, Gerd. 1995. “The Proof of Fermat’s Last Theorem by R. Taylor and A. Wiles”, *Notices of the American Mathematical Society* **42** (7) (July), 743–746; <http://www.ams.org/notices/199507/faltings.pdf>.

- FREI, Günther. 2007. "The Unpublished Section Eight: On the Way to Function Fields over a Finite Field", in Catherine Goldstein, Norbert Schappacher, & Joachim Schwermer (eds.), *The Shaping of Arithmetic: After C. F. Gauss's Disquisitiones Arithmeticae* (Berlin/Heidelberg/New York: Springer-Verlag), 159–197.
- FUSS, Paul-Heinrich. 1845. (hsg.), *Correspondance mathématique et physique de quelques célèbres géomètres du XVIIIème siècle, précédée d'une notice sur les travaux de Léonard Euler, tant imprimés qu'inédits et publiée sous les auspices de l'Académie impériale des sciences de Saint -Petersbourg* (Saint-Petersburg: Impr. de l'Académie impériale des sciences; reprinted: New York: Johnson Reprint Corp., 1968), 2 vols.; available online at: <http://math.dartmouth.edu/~euler/correspondence/>.
- GANÀ, Francesco. 1985. "Peirce e Dedekind: la definizione di insiemi finito", *Historia Mathematica* **12**, 203–218.
- GAUSS, Carl Friedrich. 1801. *Disquisitiones arithmeticae* (Leipzig: Gerhard Fleischer); English translation by Arthur A. Clarke, S.J.: (New Haven/London: Yale University Press, 1966).
- GÖDEL, Kurt. 1931. „Über die formal unentscheidbare Sätze der Principia mathematica und verwandter Systeme, I“, *Monatshefte für Mathematik und Physik* **38**, 173–198; English translation by Stefan Bauer-Mengelberg as "On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems, I", in Jean van Heijenoort (ed.), *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931* (Cambridge, Mass.: Harvard University Press, 1967), 596–616; Stuart G. Shanker (ed.), *Gödel's Theorem in Focus* (London/New York: Croom Helm, 1988), 17–47; English translation by Elliott Mendelson in Martin Davis (ed.), *The Undecidable: Basic Papers on Undecidable Propositions, Unsolvable Problems and Computable Functions* (Hewlett, N.Y.: Raven Press, 1965; reprinted: Mineola, N.Y.: Dover Publications, 2004), 4–38; reprinted, German and English on facing pages, Kurt Gödel, (Solomon Feferman, et al., eds.), *Collected Works*, vol. I: *Publications 1929–1936* (Oxford/New York: Oxford University Press), 144–195.
- GOLDBACH, Christian. 1742. Letter to Leonhard Euler, 7 June 1742; published: [Fuss 1845], I, 125–129; available online at: <http://math.dartmouth.edu/~euler/correspondence/letters/OO0765.pdf>.
- GROSSWALD, Emil. 1984. *Topics in the Theory of Numbers* (Boston/Basel/Stuttgart: Birkhäuser, 2nd ed.).
- HENRY, Charles. 1879. „Recherches sur les manuscrits de Pierre de Fermat“, *Bulletino di Bibliografica e Storia delle Scienze Matematiche e Fisiche* **XII**, 477–568, 619–740.
- HILBERT, David. 1897. „Über die Theorie der algebraischen Zahlkörper“, *Jahresbericht der Deutschen Mathematiker-Vereinigung* **4** (1897), Teil V, pp. 175–546.
- _____. 1899. „Über die Theorie des relativquadratischen Zahlkörpers“, *Mathematische Annalen* **51** (1899), 1–127, and *Göttinger Nachrichten*, 1898.
- _____. 1901. "Mathematische Probleme", *Archiv für Mathematik und Physik* (3 Reihe) **1**, 44–63, 213–237; English translation by Mary Winston Newsom: "Mathematical Problems", *Bulletin of the American Mathematical Society* **8** (1901-02), 437–479; reprinted: Felix E. Browder (ed.), *Mathematical Developments Arising from Hilbert Problems* (Providence, R. I.: American Mathematical Society, 1974), 1–34.
- HOUSER, Nathan. "Introduction" to [Peirce 2010], xxv–xcvii.
- JUSKEVIC [YUSHKEVICH], Adol'f Pavlovich, & Eduard WINTER (hsg.). 1965. *Leonhard Euler und Christian Goldbach: Briefwechsel 1729-1764* (Berlin: Akademie-Verlag).
- KETNER, Kenneth Laine Ketner, et al. (comp.). 1977. *A Comprehensive Bibliography and Index of the Published Works of Charles Sanders Peirce, with a Bibliography of Secondary Studies* (Greenwich, Connecticut: Johnson Associates, 1977; 2nd ed. Bowling Green, OH: Philosophy Documentation Center (Bowling Green, OH, 1986).
- KUMMER, Ernst Eduard. 1847a. «De numeris complexis, qui radicibus unitatis et numeris integris realibus constant», *Gratulationschrift der Universität Breslau zur Jubelfeier der Universität. Königsberg, 1844*; reprinted in *Journal de Mathématique* **12** (1844), 185–212.
- _____. 1847b. „Über die Zerlegung der aus Wurzeln der Einheit gebildeten complexen Zahlen in ihre Primfactoren“, *Journal für reine und angewandte Mathematik (Crelle)* **35**, 327–367.
- LAUBENBACHER, Reinhard C., & David J. PENNELLEY. 1994. "Gauss, Eisenstein, and the 'Third' Proof of the Quadratic Reciprocity Theorem: Eine kleines Schauspiel", *The Mathematical Intelligencer* **16** (1994), 67–72.
- LEGENDRE, Adrien-Marie. 1785. "Recherches d'analyse indéterminée", *Histoire de l'Académie royale des sciences avec les Mémoires du Mathématique et Physique pour la meme Année* (1785), 465–559; available online at: <http://gallica.bnf.fr>.
- _____. 1798. *Essai sur la théorie des nombres* (Paris: Dupont).
- LEMMERMAYER, Franz. 2000. *Reciprocity Laws from Euler to Eisenstein* (Berlin/New York: Springer Verlag).
- MATIYASEVICH, Yurii Vladimirovich [МАТИЯСЕВИЧ, Ю. В.]. 1970. "Diofantovnost' perechislimykh mnozhestv", *Doklady Akademii Nauk SSSR* **191**, 279–282; English translation: "Enumerable Sets are Diophantine", *Soviet Mathematics. Do-*

- klady. **11** (1970), 354–357.
- MATIYASEVICH, Yuri Vladimirovich. 1993a. *Desyataya problema Gil'berta* (Moscow: Nauka); English translation : [Matiyasevich 1993b].
- _____. 1993b. *Hilbert's Tenth Problem* (Cambridge, MA: MIT Press).
- NAGELL, Trygve. 1951. *Introduction to Number Theory* (Stockholm: Almqvist & Wiksell; New York: J. Wiley; 2nd ed., New York: Chelsea Publishing Co., 1964).
- ORE, Øystein, 1988. *Number Theory and Its History* (New York: Dover Publishing Co.; original edition: New York: McGraw-Hill, 1948).
- PEANO, Giuseppe. 1889. *Arithmetices principia, nova methodo exposita* (Torino: Bocca).
- PEIRCE, Charles Sanders. 1880/81. "The Axioms of Number"; Robin Catalog MS#380; Winter 1880-1881; published: [Peirce 1989], 222–224.
- _____. 1881a. "On the Logic of Number", *American Journal of Mathematics* **4**, 85–95; reprinted: [Peirce 1989], 299–309.
- _____. 1881b. [On the Logic of Number], untitled; ms, MS# 393, summer 1881; marginalia to [Peirce 1881a].
- _____. 1881c. "Continuous, simple quantity"; ms, MS# 393, summer 1881.
- _____. ca. 1881?. "Axioms of Number"; Robin Catalog, No. 40. A. MS., n.p., [ca.1881?], 4 pp.
- _____. 1886. "Fundamental Properties of Number"; Robin catalog MS#567, 5 January 1886; published: (Christian J. W. Kloesel, ed.). *Writings of Charles S. Peirce: A Chronological Edition*, vol. 5: 1884–1886 (Bloomington/Indianapolis: Indiana University Press, 1993), 283–284.
- _____. ca. 1887a. "Number"; ms., ca. 1887; published: [Peirce 2000], 155.
- _____. ca. 1887b. "Logic of Number", ms., ca. 1887; published: [Peirce 2000], 156–163.
- _____. ca. 1890. "Logical Studies of the Theory of Numbers", ca. 1890; published: [Peirce 2010], 55–56.
- _____. 1889-1910a. "Law of Reciprocity of prime numbers", in [Whitney 1889], vol. IV, 3376.
- _____. 1889-1910b. "Number", in [Whitney 1889], vol. V, 4038.
- _____. 1889-1910c. "Quadratic reciprocity", in [Whitney 1889], vol. VI, 4884.
- _____. 1905-6. "Analysis of some Demonstrations concerning definite Positive Integers"; Robin Catalog. No. 70. (N)A. MS., G-1905-6, pp. 1–20, with 50pp. of variants and notes.
- _____. n.d. "Numerical Equations"; Robin Catalog, No. 69. MS., n.p., n.d., 1 folded sheet (2 pp.).
- _____. 1989. (Christian J. W. Kloesel, ed.). *Writings of Charles S. Peirce: A Chronological Edition*, vol. 4: 1879–1884 (Bloomington/Indianapolis: Indiana University Press).
- _____. 2000. (Nathan Houser, ed.), *Writings of Charles S. Peirce: A Chronological Edition*, vol. 6: 1886–1890 1886 (Bloomington/Indianapolis: Indiana University Press).
- _____. 2010. (Nathan Houser, ed.), *Writings of Charles S. Peirce: A Chronological Edition*, vol. 8: 1890–1892 (Bloomington/Indianapolis: Indiana University Press).
- PIEPER, Herbert. 1997. „Über Legendres Versuch, das quadratische Reziprozitätsgesetz zu beweisen“, *Natur, Mathematik und Geschichte. Acta Hist. Leopold. No. 27*, 223–237.
- RIEMANN, Geoerg Friedrich Bernhard. 1859. „Über die Anzahl der Primzahlen unter einer gegebenen Grösse“, *Monatsbericht der Königlich Preussische Akademie der Wissenschaften zu Berlin* (Nov. 1859), 671–680; reprinted: Hermann Weyl (ed.), *Das Kontinuum und Andere Monographien* (New York: Chelsea, 1972); in English translation: Harold M. Edwards, *Riemann's Zeta Function* (New York: Dover Publications, 2001), „Appendix“, 299–305.
- SHAFAREVICH, Igor Rostislavovich [ШАФАРЕВИЧ, И. П.]. 1950. "Obshchei zakon vzaimnosti", *Matematicheskii Sbornik* (N.S.) **26(68)**, 113–146.
- SHIELDS, Paul Bartram. 1981. *Charles S. Peirce on the Logic of Number* (Ph.D. thesis, Fordham University).
- _____. 1997. "Peirce's Axiomatization of Arithmetic", in Nathan Houser, Don D. Roberts, & James Van Evra (eds.), *Studies in the Logic of Charles Sanders Peirce*. Indianapolis/Bloomington: Indiana University Press, 1997), 43–52.
- TANNERY, Paul. 1883. "Sur la date des principales découvertes de Fermat", *Bulletin des Sciences Mathématiques et Astronomiques* (2) **VII**, 116–128.
- TAYLOR, Richard, & Andrew WILES. 1995. "Ring Theoretic Properties of Certain Hecke Algebras", *Annals of Mathematics* (2) **141**, 553–572.
- WEINTRAUB, Steven H. 2009. "On Legendre's Work on the Law of Quadratic Reciprocity"; preprint; electronic version available at: <http://www.lehigh.edu/~shw2/preprints/legendre.pdf>.
- WHITNEY, William Dwight (ed.). 1889. *The Century Dictionary: An Encyclopedic Lexicon of the English Language* (New York: The Century Co.; revised ed., 1895).
- WILES, Andrew. 1995. "Modular Elliptic Curves and Fermat's Last Theorem", *Annals of Mathematics* (2) **141**, 443–551; <http://math.stanford.edu/~lekheng/flt/wiles.pdf>.

YOUNG, Jacob William Albert. 1894. "Bachmann's Theory of Numbers", review of [Bachmann 1892], *Bulletin of the New York Mathematical Society* **3**, no. 9 (June), 215–222.

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